

Triangle-free induced subgraphs of polarity graphs

Jared Loucks*

Craig Timmons†

Abstract

Given a finite projective plane Π and a polarity θ of Π , the corresponding polarity graph is the graph whose vertices are the points of Π . Two distinct vertices p and p' are adjacent if p is incident to $\theta(p')$. Polarity graphs have been used in a variety of extremal problems, perhaps the most well-known being the Turán number of the cycle of length four. We investigate the problem of finding the maximum number of vertices in an induced triangle-free subgraph of a polarity graph. Mubayi and Williford showed that when Π is the projective geometry $PG(2, q)$ and θ is the orthogonal polarity, an induced triangle-free subgraph has at most $\frac{1}{2}q^2 + O(q^{3/2})$ vertices. We generalize this result to all polarity graphs, and provide some interesting computational results that are relevant to an unresolved conjecture of Mubayi and Williford.

1 Introduction

Let $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a finite projective plane. A polarity θ of Π is a bijection of order two that maps \mathcal{P} to \mathcal{L} , maps \mathcal{L} to \mathcal{P} , and has the property that for any point p and line l ,

$$p\mathcal{I}l \text{ if and only if } \theta(l)\mathcal{I}\theta(p).$$

Polarities in projective planes have a rich history in finite geometry. For further discussion, we recommend Hughes and Piper ([13], Chapter 12) or Dembowski ([6], Chapter 3). Given a finite projective plane Π and a polarity θ of Π , the corresponding *polarity graph*, denoted $G(\Pi, \theta)$, is the graph whose vertex set is \mathcal{P} . Two distinct vertices p and p' are adjacent if and only if $p\mathcal{I}\theta(p')$.

Let q be a power of a prime. The special case when the plane Π is $PG(2, q)$ and θ is the polarity that maps a subspace to its orthogonal complement appears frequently in combinatorics. This graph, which we denote by ER_q , was introduced to the graph theory community by Erdős, Rényi [8], Brown [4], and Erdős, Rényi, and Sós [9]. Since then, ER_q has appeared in many different contexts such as Ramsey theory, spectral and structural graph theory, and Turán problems. For instance, ER_q has the maximum

*Department of Mathematics and Statistics, California State University Sacramento, jaredloucks@csus.edu

†Department of Mathematics and Statistics, California State University Sacramento, craig.timmons@csus.edu. Research supported in part by Simons Foundation Grant #359419.

number of edges among all graphs with $q^2 + q + 1$ vertices that have no cycle of length four. This was proved by Füredi [10] and is one of the most important results concerning bipartite Turán problems. In fact, any polarity graph $G(\Pi, \theta)$ where Π has order q has this same property provided that the number of absolute points of θ is $q + 1$. Such a polarity is called *orthogonal*. A classical result of Baer [3] states that any polarity of a projective plane of order q has at least $q + 1$ absolute points. Thus, orthogonal polarities are the ones that have the fewest number of absolute points.

A consequence of its significance in graph theory is that different properties of ER_q have been studied. In [8, 9] it is shown that ER_q has $\frac{1}{2}q^2(q + 1)$ edges, has diameter 2, and does not contain a cycle of length four. In general, this is true for any polarity graph $G(\Pi, \theta)$ for which θ is orthogonal. The automorphism group of ER_q was determined by Parsons [18], and then again by Bachratý and Širáň [2] who provided simpler proofs. The independence number and chromatic number of ER_q was studied in [11, 12, 17, 20] and [19], respectively.

In this note, we consider the following problem of Mubayi and Williford [17].

Problem 1.1 *Determine the maximum number of vertices in an induced subgraph of ER_q that contains no cycle of length three.*

One of the motivations behind Problem 1.1 comes from Turán theory. Let us write $\text{ex}(n, \{C_3, C_4\})$ for the maximum number of edges in an n -vertex graph with no cycle of length 3 or 4. Note that such a graph has girth at least 5. The incidence graph of a projective plane has girth at least 5. Erdős [7] has conjectured that this construction is asymptotically best possible; that is

$$\text{ex}(n, \{C_3, C_4\}) = \frac{1}{2\sqrt{2}}n^{3/2} + o(n^{3/2}) \quad ? \quad (1)$$

It was recently conjectured by Allen, Keevash, Sudakov, and Verstraëte [1] that (1) can be improved. More precisely, if $z(n, C_4)$ is the maximum number of edges in an n -vertex bipartite graph with no cycle of length 4, then Allen et. al. conjecture that

$$\liminf_{n \rightarrow \infty} \frac{\text{ex}(n, \{C_3, C_4\})}{z(n, C_4)} > 1 \quad ?$$

The best known lower bound on $\text{ex}(n, \{C_3, C_4\})$ comes from an induced triangle free subgraph of ER_q and shows that for infinitely many n ,

$$\text{ex}(n, \{C_3, C_4\}) > z(n, C_4) + \frac{1}{8}n + O(\sqrt{n}).$$

This construction is due to Parsons [18] and will be discussed momentarily.

Let us now return to Problem 1.1. Mubayi and Williford [17] showed that for any q , the maximum number of vertices in an induced triangle-free subgraph of ER_q is at most

$$\frac{1}{2}q^2 + q^{3/2} + O(q).$$

Using an approach based on finite geometry, we generalize this upper bound to all polarity graphs.

Theorem 1.2 *Let Π be a projective plane of order q , θ be a polarity of Π , and $G(\Pi, \theta)$ be the corresponding polarity graph. If H is an induced triangle-free subgraph of $G(\Pi, \theta)$, then*

$$|V(H)| \leq \frac{1}{2}(q^2 + q + 1) + \sqrt{q} \left(\frac{q^2 + q + 1}{q + 1} \right).$$

As for lower bounds, Parsons [18] showed that when q is a power of an odd prime, ER_q contains an induced triangle-free subgraph on $\binom{q}{2}$ vertices if $q \equiv 1 \pmod{4}$, and on $\binom{q+1}{2}$ vertices if $q \equiv 3 \pmod{4}$. By the above mentioned result of Mubayi and Williford [17], the construction of Parsons is asymptotically best possible. The following was conjectured in [17] and asserts that one cannot do better than Parsons' construction.

Conjecture 1.3 (Mubayi, Williford [17]) *Let q be a power of an odd prime. The maximum number of vertices in an induced triangle-free subgraph of ER_q containing no absolute points is $\binom{q}{2}$ if $q \equiv 1 \pmod{4}$, and $\binom{q+1}{2}$ if $q \equiv 3 \pmod{4}$.*

We remark that the reason for excluding absolute points is that in any polarity graph, a vertex that is an absolute point will not lie in a triangle. We prove this in the next section and it is a known result.

Our computational results show that if Conjecture 1.3 is true, then one must assume some lower bound on q as the conjecture fails for small values of q . These new lower bounds are summarized in the following table where we write $f(ER_q)$ for the maximum number of vertices in an induced triangle-free subgraph of ER_q that contains no absolute points. Those values marked with a * indicate an improvement over Parsons' construction.

q	$f(ER_q)$
3	= 6
5*	= 16
7*	≥ 30
9*	≥ 46
11	≥ 66
13*	≥ 80

For comparison with Conjecture 1.3, our lower bound for 7, 9, and 13 exceeds the conjectured bound by 2, 10, and 2, respectively. The lower bound for 5 was done by a simple brute force search argument but for larger q , such a search is impossible. A Mathematica [21] notebook file giving these lower bounds is available on the second listed author's website [15].

When q is a power of 2, Mattheus, Pavese, and Storme [16] recently proved that ER_q contains an induced subgraph of girth at least 5 with $\frac{q(q+1)}{2}$ vertices. This answers a question of Mubayi and Williford [17]. Another polarity graph of interest is the unitary polarity graph U_q . If q is an even power of a prime, the graph U_q has the same vertex set as ER_q . Let us write (x_0, x_1, x_2) for a vertex in ER_q where (x_0, x_1, x_2) is a nonzero

vector, and two 3-tuples represent the same vertex if one is a nonzero multiple of the other. Two distinct vertices (x_0, x_1, x_2) and (y_0, y_1, y_2) are adjacent if

$$x_0^{\sqrt{q}} y_0 + x_1^{\sqrt{q}} y_1 + x_2^{\sqrt{q}} y_2 = 0.$$

Despite this relatively simple algebraic condition for adjacency, we were unable to find a triangle-free induced subgraph of U_q with $\frac{1}{2}q^2 - o(q^2)$ vertices. In general, we conclude our introduction with the following question which generalizes one asked in [17].

Question 1.4 *Given a projective plane Π of order q and a polarity θ of Π , is it always possible to find a triangle-free subgraph of $G(\Pi, \theta)$ with $\frac{1}{2}q^2 - o(q^2)$ vertices?*

The rest of this note is organized as follows. In Section 2 we prove Theorem 1.2. In Section 3 we discuss some of our computational results and make some additional remarks.

2 Proof of Theorem 1.2

Throughout this section, Π is a projective plane of order q , θ is a polarity of Π , and $G(\Pi, \theta)$ is the corresponding polarity graph.

The first lemma is known but a proof is included for completeness.

Lemma 2.1 *No absolute point of θ is in a triangle in $G(\Pi, \theta)$.*

Proof. Suppose p_1 is an absolute point that lies in a triangle and the other vertices of the triangle are p_2 and p_3 . It must be the case that all three of p_1 , p_2 , and p_3 are incident to $\theta(p_1)$. However, p_1 is incident to $\theta(p_3)$ and p_2 is incident to $\theta(p_3)$. As the line through any pair of points is unique, $\theta(p_1) = \theta(p_3)$ which implies $p_1 = p_3$, a contradiction. ■

Lemma 2.2 *If p is a vertex of $G(\Pi, \theta)$ and p is not an absolute point of θ , then the vertices adjacent to p can be partitioned into two sets A_p and B_p such that*

1. *the set A_p is a (possibly empty) subset of the absolute points of θ , and*
2. *the vertices in B_p induce a matching in $G(\Pi, \theta)$, and no vertex in B_p is an absolute point of θ .*

Proof. Since Π has order q , there are exactly $q+1$ lines that p is incident to. These lines can be written as $\theta(p_1), \theta(p_2), \dots, \theta(p_{q+1})$ for some $p_1, p_2, \dots, p_{q+1} \in \mathcal{P}$. By definition, we have that p is adjacent to p_1, p_2, \dots, p_{q+1} in the graph $G(\Pi, \theta)$. Note that no p_i is equal to p since p is not an absolute point. By relabeling if necessary, we may assume that p_1, p_2, \dots, p_c are not absolute points, and that $p_{c+1}, p_{c+2}, \dots, p_{q+1}$ are absolute points. Let $A_p = \{p_{c+1}, p_{c+2}, \dots, p_{q+1}\}$ and $B_p = \{p_1, p_2, \dots, p_c\}$. We have that A_p is a subset of the absolute points and that B_p contains no absolute points. To finish the proof of the lemma, we must show that the vertices in B_p induce a matching.

Let $p_i \in B_p$ so p is incident to $\theta(p_i)$. There is exactly one line $l \in \mathcal{L}$ such that p and p_i are both incident to l . There must be a $j \in \{1, 2, \dots, q+1\}$ such that $l = \theta(p_j)$ and so p, p_i , and p_j form a triangle. By Lemma 2.1, p_j cannot be an absolute point so $j \in \{1, 2, \dots, c\}$. If $j = i$, then p_i is an absolute point, but $p_i \in B_p$ and B_p contains no absolute points. Therefore, $j \neq i$ and the vertices p, p_i , and p_j are all distinct. Because there is exactly one line l with both p and p_i incident to l , p_i uniquely determines p_j and so the vertices in B_p induce a matching. ■

The next result is the well-known Expander Mixing Lemma.

Theorem 2.3 *Let G be a d -regular graph, possibly with loops where a loop adds one to the degree of a vertex. If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of G and $\lambda = \max_{2 \leq i \leq n} |\lambda_i|$, then for any sets $X, Y \subseteq V(G)$,*

$$\left| e(X, Y) - \frac{d|X||Y|}{n} \right| \leq \lambda \sqrt{|X||Y|}$$

where $e(X, Y) = |\{(x, y) \in X \times Y : \{x, y\} \in E(G)\}|$.

Let $G^\circ(\Pi, \theta)$ be the graph obtained from $G(\Pi, \theta)$ by adding one loop to each absolute point. It is known that the eigenvalues of $G^\circ(\Pi, \theta)$ are $q+1$ with multiplicity 1, and all others have magnitude at most \sqrt{q} . For any subset of vertices $J \subseteq V(G^\circ(\Pi, \theta))$, we have by Theorem 2.3,

$$\left| e(J, J) - \frac{(q+1)|J|^2}{q^2 + q + 1} \right| \leq \sqrt{q}|J|.$$

We now have all of the tools that we need in order to prove Theorem 1.2.

Proof of Theorem 1.2. Let $J \subset V(G(\Pi, \theta))$ and assume that J contains no absolute points, and that the subgraph induced by J contains no triangles. Since J contains no absolute points, the number of edges in $G(\Pi, \theta)$ whose endpoints are in J is the same as the number of edges in $G^\circ(\Pi, \theta)$ whose endpoints are in J . By Theorem 2.3,

$$e(J, J) \geq \frac{(q+1)|J|^2}{q^2 + q + 1} - \sqrt{q}|J|. \quad (2)$$

Note that $e(J, J) = \sum_{v \in J} d_J(v)$, where $d_J(v)$ is the number of neighbors of v in J . Let $v \in J$. By Lemma 2.2, since J contains no absolute points, all of the vertices adjacent to v are contained in B_v , and therefore induce a matching in $G(\Pi, \theta)$. Since J contains no triangles, $d_J(v) \leq \frac{|B_v|}{2} \leq \frac{q+1}{2}$. Combining this inequality with (2), we get that

$$\frac{(q+1)|J|^2}{q^2 + q + 1} - \sqrt{q}|J| \leq e(J, J) = \sum_{v \in J} d_J(v) \leq |J| \left(\frac{q+1}{2} \right).$$

Solving this inequality for $|J|$ yields

$$|J| \leq \frac{1}{2}(q^2 + q + 1) + \sqrt{q} \left(\frac{q^2 + q + 1}{q + 1} \right)$$

completing the proof of the theorem. ■

3 Concluding Remarks

We begin this section by giving a brief description of how our computational results were obtained. A close look at the proof of Theorem 1.2 suggests that one way to find a large set J that induces a triangle-free graph is to choose an independent set I of size q , and then for each vertex $v \in I$, we choose one vertex from each triangle in the neighborhood v and put it into J . This would give a set of size about $\frac{1}{2}q^2$ which is the size we are aiming for, but of course we need to avoid triangles. This is the main difficulty. Our lower bounds for $q \geq 7$ are, more or less, obtained by following this approach. More details are provided in [15].

In any polarity graph $G(\Pi, \theta)$, the neighborhood of a vertex induces a graph of maximum degree 1, otherwise we find a cycle of length four. If Π has order q , then this provides a trivial lower bound of $\frac{1}{2}(q+1)$ on the number of vertices in an induced triangle-free subgraph but there may be absolute points in this set. Regardless, this lower bound can be improved by considering the hypergraph $\mathcal{H}(\Pi, \theta)$ whose vertex set is the vertices of $G(\Pi, \theta)$ that are not absolute points. The edges of $\mathcal{H}(\Pi, \theta)$ are the triangles in $G(\Pi, \theta)$. Since a polarity has at most $q^{3/2} + 1$ absolute points (see [13]), $\mathcal{H}(\Pi, \theta)$ has at least $q^2 + q - q^{3/2}$ vertices. Furthermore, each vertex in $G(\Pi, \theta)$ is in at most $\frac{q+1}{2}$ triangles and no two triangles share an edge. This implies that $\mathcal{H}(\Pi, \theta)$ has maximum degree $\frac{q+1}{2}$ and maximum codegree 1. By a result of Duke, Lefmann, and Rödl [5], there is positive constant c , not depending on Π or θ , such that the independence number of $\mathcal{H}(\Pi, \theta)$ is at least $cq^{3/2}\sqrt{\log q}$. By definition, such a set induces a triangle-free graph in $G(\Pi, \theta)$. This argument was pointed out to the second author by Jacques Verstraëte.

In the search for induced triangle-free graphs, a related problem arose. Consider the graph ER_q where q is a power of an odd prime. The vertices of ER_q can be partitioned into three sets: the absolute points, the vertices that are adjacent to at least one absolute point, and the vertices that are not adjacent to any absolute points. This is proved in [18] and [20]. Let us call these sets A_q , S_q , and E_q , respectively. When $q \equiv 1 \pmod{4}$, the subgraph induced by E_q is triangle-free, and when $q \equiv 3 \pmod{4}$, the subgraph induced by S_q is triangle-free. This is the construction of Parsons [18] which shows Conjecture 1.3, if true, would be best possible. One can ask if this property characterizes $PG(2, q)$. That is, suppose $G(\Pi, \theta)$ is a polarity graph for which the vertex set admits a partition into three sets consisting of the absolute points of θ , the neighbors of the absolute points (which we denote by S), and the vertices not adjacent to absolute points (which we denote by E). If the subgraph induced by S or by E is triangle-free, then must $\Pi = PG(2, q)$ and θ be an orthogonal polarity of $PG(2, q)$?

References

- [1] P. Allen, P. Keevash, B. Sudakov, J. Verstraëte, Turán numbers of bipartite graphs plus an odd cycle, *J. Combin. Theory Ser. B* 106 (2014), 134–162.
- [2] M. Bachratý, J. Širáň, *Polarity graphs revisited*, *Ars Math. Contemp.* 8 (2015), no. 1, 55–67.

- [3] R. Baer, Polarities in finite projective planes, *Bull. Amer. Math. Soc.* **52**, (1946). 77–93.
- [4] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** 1966 281–285.
- [5] R. Duke, H. Lefmann, V. Rödl, On uncrowded hypergraphs, *Random Structures Algorithms* 6 (1995), no. 2-3, 209–212.
- [6] P. Dembowski, *Finite Geometries*, Springer-Verlag Berlin Heidelberg, Germany, 1968.
- [7] P. Erdős, Some recent progress on extremal problems in graph theory, *Congr. Numer.* **14** (1975), 3–14.
- [8] P. Erdős, A. Rényi, On a problem in the theory of graphs. (Hungarian) *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** 1962 623–641 (1963).
- [9] P. Erdős, A. Rényi, V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** 1966 215–235.
- [10] Z. Füredi, On the number of edges of quadrilateral-free graphs, *J. Combin. Theory Ser. B* 68 (1996), no. 1, 1–6.
- [11] C. Godsil, M. Newman, Eigenvalue bounds for independent sets, *J. Combin. Theory Ser. B* 98 (2008), no. 4, 721–734.
- [12] S. Hobart, J. Williford, The independence number for polarity graphs of even order planes, *J. Algebraic Combin.* 38 (2013), no. 1, 57–64.
- [13] D. R. Hughes, F. C. Piper, *Projective Planes*, GTM Vol. 6, Springer-Verlag New-York-Berlin, 1973.
- [14] F. Lazebnik, J. Verstraëte, On hypergraphs of girth five, *Electron. J. Combin.* 10 (2003), #R25.
- [15] J. Loucks, C. Timmons, Supporting Mathematica notebook file available at <http://webpages.csus.edu/craig.timmons/papers>
- [16] S. Mattheus, F. Pavese, L. Storme, On the independence number of graphs related to a polarity, arXiv:1704.00487v1 3 Apr 2017.
- [17] D. Mubayi, J. Williford, On the independence number of the Erdős-Rényi and projective norm graphs and a related hypergraph, *J. Graph Theory* 56 (2007), no. 2, 113–127.
- [18] T. D. Parsons, Graphs from projective planes, *Aequationes Math.* 14 (1976), no. 1-2, 167–189.

- [19] X. Peng, M. Tait, C. Timmons, On the chromatic number of the Erdős-Rényi orthogonal polarity graph, *Electron. J. Combin.* 22 (2015), no. 2, Paper 2.21, 19 pp.
- [20] J. Williford, *Constructions in finite geometry with applications to graphs*, PhD Thesis, University of Delaware, 2004.
- [21] Wolfram Research, Inc., Mathematica, Version 11.0, Champaign, IL (2016).